The reverse Wagner/Within model with remanufacturing and manufacturing options. Exact results and holding cost stability regions

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Abstract

In this paper, we study a reverse Wagner/Whitin production and inventory control model. In such, reverse (product recovery) models, used products are returned back and stored for possible future remanufacturing. The model supposes that cost and demand parameters are constant over time and a sufficiently large quantity of used products is available at the beginning of the planning horizon. We consider policies that take a given number of set up for remanufacturing and manufacturing respectively. In this class of policies we find the optimal policy, which specifies the number of periods where demand is covered either only by remanufactured items or by newly manufactured items respectively, the periods where remanufacturing or manufacturing activities take place and the corresponding lot sizes. Further, we construct stability regions for the optimal policy, which are expressed as intervals of the ratio of holding cost parameters.

Keywords: Remanufacturing, manufacturing, stability, inventory, product recovery

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1. Introduction

Increasing environmental consciousness of customers and product take back regulations of governments offer the possibility to satisfy customer demand from recovered used products, instead of exclusively manufacturing new products and disposing of all returned products. Inventory models for these product recovery systems, share several features with two-supplier inventory models in that product returns represent a second mode of supply in addition to the production alternative. The European working group on reverse logistics REVLOG [5] defines four types of recovery: direct reuse, remanufacturing, recycling and incineration. In the specific model that we analyze in this paper, only remanufacturing is considered. REVLOG defines remanufacturing as the process consisting of disassembly, cleaning, testing, re-assembly etc., after which a product is as-good-as-new.

Since remanufactured products are as-good-as-new, they can be used to satisfy demand for new products. For products that have not yet reached the end of their life-cycle, there are typically more demands than returns. Hence, at least some demand still has to be satisfied by manufactured (new) products. Such situations with both manufacturing and remanufacturing, lead to interesting questions from an inventory control point of view. When should products be manufactured or remanufactured and in what quantities? When should returned products be disposed of?

In recent years, a number of authors have studied these questions. Laan et al. [3] and also Inderfurth [2] studied review policies for production planning and inventory control in stochastic manufacturing/ remanufacturing/ disposal systems. Richter [8] applies the EOQ model to a similar deterministic manufacturing/ remanufacturing/ disposal system. Richter and Sombrutzki [7] in their pioneering article studied also models, known now, as reverse Wangner/Within dynamic lot sizing models. One characteristic of these models is the constant set up costs for remanufacturing and manufacturing. Using suitable transformations, they managed to transform these recovery models to models of the Wagner-Whitin type. They then proved the zero-inventory-property for the optimal solution and solved them by the Wagner-Whitin algorithm [11]. In a subsequent paper, Richter and Weber [6] extended the reverse Wagner/Whitin type models by introducing variable manufacturing and remanufacturing cost and for the case of time-constant cost and demand data, they proved the optimality of a policy starting with remanufacturing before switching to manufacturing. Closing this article, the authors speculated that the techniques used by

Papachristos and Ganas [4] and Chand [1] could be used to solve some of their proposed models and especially to construct stability regions for the obtained solutions. This speculation motivated the research presented in this paper.

In this paper, we consider a periodic review single-product recovery system over a finite horizon. At the beginning of the horizon, we suppose that a sufficiently large quantity of used products with low inventory cost is available. Demand at every period is constant and is satisfied by remanufactured and/or newly manufactured products. The so created stock is called stock of final products. Set up manufacturing /remanufacturing costs and holding costs for used and final products are constant while backlogging is not permitted. For this problem it is known [6] that, an optimal policy is characterized by the following property. At the initial periods of the horizon and up to some period, called switching period, demand is satisfied only by remanufacturing. At the switching period remanufacturing activities stop and demand for the next periods is satisfied only by manufacturing new items. We consider policies that take a given number of set up for remanufacturing and manufacturing respectively. In this class of policies we find the optimal policy which specifies: the number of periods where demand is covered either only by remanufacturing items or by manufacturing new items respectively, the periods where remanufacruring or manufacturing activities will take place and the corresponding lot sizes. Continuing we come to the other objective of the paper, which is the construction of stability regions for the optimal policy, which in our case are expressed as intervals of the ratio of holding cost parameters.

The paper is organized as follows: the second section contains the mathematical formulation of the problem. In the third section we present results concerning the structure of the optimal policy. For any switching period and numbers of set up for remanufacturing/manufacturing the minimum cost for the problem is analytically expressed as a function of these three parameters. This function is proved to be convex with respect to the switching period. In the fourth section we propose an algorithm which computes the optimal policy and constructs stability regions for any combination of the holding cost parameters. The fifth section contains numerical examples which illustrate the application of all results presented in the paper and especially explains the application of the algorithm. Concluded remarks and proposals for further research are given in the final, sixth section.

2. Problem formulation

We consider a single-item periodic review inventory recovery system over a finite horizon T, having the following operative characteristics:

1. At the beginning of the first period, a quantity $d(d \geq TD)$ of used products (returned products) is in stock (recoverable inventory) waiting for remanufacturing.
2. There are no further returns of used products in periods $t = 2,3,...,T$, i.e. $d_t = 0, t = 2,3,...,T$.
3. Demand D, at every period is constant and is satisfied either by remanufactured products and/or by newly manufactured ones.
4. Remanufactured and newly produced products are considered to have the same quality and value for the customer.
5. Set up cost S for making new products and R for remanufacturing used ones are constant and independent of the quantity.
6. Holding costs H for final products (the term describes remanufactured and newly manufactured items) and h for used products are constant at every period. They are charged to the end of period stock and we suppose that $h < H$.
7. Shortages are not allowed.
8. The planning horizon is composed of T discrete time periods of equal length.
9. Used products, which have not remanufactured, are kept in a recoverable inventory store until the end of horizon.

Additional notations, which will be used subsequently, are the following:

x_t the lot size of used products remanufactured at the beginning of period t

z_t the lot size of new products produced at the beginning of period t

I_t the inventory level of final products at the end of period t

y_t the inventory level of used products at the end of period t

n_1 the number of remanufacturing set up

n_2 the number of manufacturing set up

N_k the set $\{n_1, n_1+1,...,T-n_2\}$

k the switching period (number of periods that demand is covered only by remanufactured items), $k \in N_k = \{n_1, n_1+1,...,T-n_2\}$

$P(n_1,n_2,k)$ the set of policies covering demand for the first k periods, k given, by taking n_1 remanufacturing set up and the rest $T-k$ periods by taking n_2 manufacturing set up

$P(n_1,n_2,X)$ the set of policies covering demand for the first $x \in X$ periods by taking n_1 remanufacturing set up and the rest $T-x$ by taking n_2 manufacturing set up, X is a subset of N_k

$\lceil x \rceil$ the smallest integer greater than or equal to x

$\lfloor x \rfloor$ the largest integer less than or equal to x

$f(x_t)$ $\begin{cases} 1, & if\ x_t > 0 \\ 0, & if\ x_t = 0 \end{cases}$

The cost for period t of such a system is $Rf(x_t)+Sf(z_t)+hy_t+HI_t$. The problem is to find the x_t, z_t which minimize the total cost and has the following formulation:

$$(P) \qquad \min_{x_t,z_t} \sum_{t=1}^{T} \left(Rf(x_t)+Sf(z_t)+hy_t+HI_t \right) \tag{1}$$

$$\text{s.t.} \quad \begin{aligned} & y_1 = y_0 - x_1 + d = d - x_1 \\ & y_t = y_{t-1} - x_t && t=2,3,\ldots,T \\ & I_t = I_{t-1} + x_t + z_t - D && t=1,2,\ldots,T \\ & \sum_{t=1}^{T} f(x_t) = n_1, \ \sum_{t=1}^{T} f(z_t) = n_2 \\ & y_0 = I_0 = I_T = 0 \\ & x_t, y_t, z_t, I_t \geq 0 && t=1,2,\ldots,T \end{aligned} \tag{2}$$

Additionally to this we are interested to study the stability issue for the obtained solution.

3. Searching for the optimal policy

It is obvious from the introduction, that research on these reverse Wagner/ Within type models is really very limited. To the best of our knowledge there is no other work on this topic except the two papers by Richter and Weber [6] and Richter and Sombrutzki [7]. On the other hand, work on stability issues in lot sizing problems is not much better. We cite here the two papers by Richter [9] and Voros [10] which we consider as the most important contributions in the field.

Richter and Sombrutzki [7] have studied various models of the type considered above. Using a simple but very clever transformation they transformed their models to models of the Wagner/Within type. In this transformed form they proved the known as "zero-inventory property" which for their case is mathematically expressed by the following equations:

$$\begin{aligned} &x_t z_t = 0 && t = 1,2,\dots,T \\ &I_{t-1}(x_t + z_t) = 0 && t = 1,2,\dots,T. \end{aligned} \tag{3}$$

The first equation ensures that, at every period t at most one of the remanufacturing/manufacturing activities may take place, while the second ensures that this will occur only if the inventory level of final products at the end of previous period is zero. This key result enabled them to use the W-W algorithm to solve their models. In a subsequent paper Richter and Weber [6] extended the above work by introducing variable costs for the remanufacturing/manufacturing activities. In the case of a model with constant demand and cost parameters and a large quantity of used products available at the beginning of the horizon, they proved that an optimal policy covers the first k periods of the horizon only by remanufactured items and the remaining ones only with newly produced items. So, at period $k+1$ production switches from remanufacturing to manufacturing and this k fulfils the inequality

$$t_p < k+1 \le t_r + i_r,$$

where: $t_p = \left\lfloor T+1-\frac{R-S}{h} \right\rfloor$, $\quad t_r = \left\lceil T+1-\sqrt{\frac{R-S}{h}} \right\rceil$, $\quad i_r = \left\lfloor \frac{R}{H-h} \right\rfloor$.

For any $k \in K = \{k:\ t_p - 1 < k \le t_r + i_r - 1\}$ they found the minimum cost applying the W-W algorithm and then compared these costs to locate the overall optimum k. In this paper k will be referred as a switching period.

3.1. Optimal policy in the set of policies $P(n_1, n_2, k)$

For given k problem P is decomposed into two subproblems. A pure remanufacturing for periods $1,2,\dots,k$ and a pure manufacturing for periods $k+1, k+2,\dots,T$. These are:

$$(P_1) \qquad \min_{x_t} \sum_{t=1}^{k} \left(Rf(x_t) + hy_t + HI_t \right) \tag{4}$$

$$\text{s.t.}\quad \begin{aligned} & y_1 = d - x_1 \\ & y_t = y_{t-1} - x_t \qquad t = 2,3,\dots,k \\ & I_t = I_{t-1} + x_t - D \quad t = 1,2,\dots,k \\ & \sum_{t=1}^{k} f(x_t) = n_1 \\ & I_0 = I_k = 0 \\ & x_t, y_t, I_t \geq 0 \qquad t = 1,2,\dots,k \end{aligned} \tag{5}$$

$$\text{and} \quad (P_2) \qquad \min_{z_t} \sum_{t=k+1}^{T} \left(Sf(z_t) + hy_t + HI_t \right) \tag{6}$$

$$\text{s.t.}\quad \begin{aligned} & y_t = y_k \\ & I_t = I_{t-1} + z_t - D \\ & \sum_{t=k+1}^{T} f\left(z_t\right) = n_2 \\ & I_k = I_T = 0 \\ & z_t, y_t, I_t \geq 0, \qquad t = k+1, k+2, \dots, T \end{aligned} \tag{7}$$

The realization of stock levels for used and final products for a problem with $T = 10$, $n_1 = 3$, $n_2 = 2$ and $k = 7$, are given in Figures 1 and 2.

Summing the three inventory equations of (5) we obtain:

$$I_t + y_t = d - tD, \qquad t = 1,2,\dots,k. \tag{8}$$

Using this relation, the holding cost in the objective function (4) is written as:

$$hy_t + HI_t = h(y_t + I_t) + (H - h)I_t = h(d - tD) + (H - h)I_t = h(d - tD) + H^{'}I_t, \quad t = 1,2,\dots k$$

with $H^{'} = H - h$.

So, the objective function becomes:

$$\sum_{t=1}^{k} \left(Rf(x_t) + H^{'}I_t + h(d - tD) \right) = \sum_{t=1}^{k} \left(Rf(x_t) + H^{'}I_t \right) + h(kd - \frac{k(k+1)}{2}D)$$

and the pure remanufacturing problem is transformed to its equivalent:

$$(P_1^{'}) \qquad \min_{x_t} \sum_{t=1}^{k} \left(Rf(x_t) + H^{'}I_t \right) + W \tag{9}$$

$$\text{s.t.}\quad \begin{aligned} & I_t = I_{t-1} + x_t - D \quad t = 1,2,\dots,k \\ & \sum_{t=1}^{k} f(x_t) = n_1 \\ & I_0 = I_k = 0 \\ & x_t, I_t \geq 0 \qquad t = 1,2,\dots,k. \end{aligned} \tag{10}$$

where $W = khd - \frac{k(k+1)}{2}hD$.

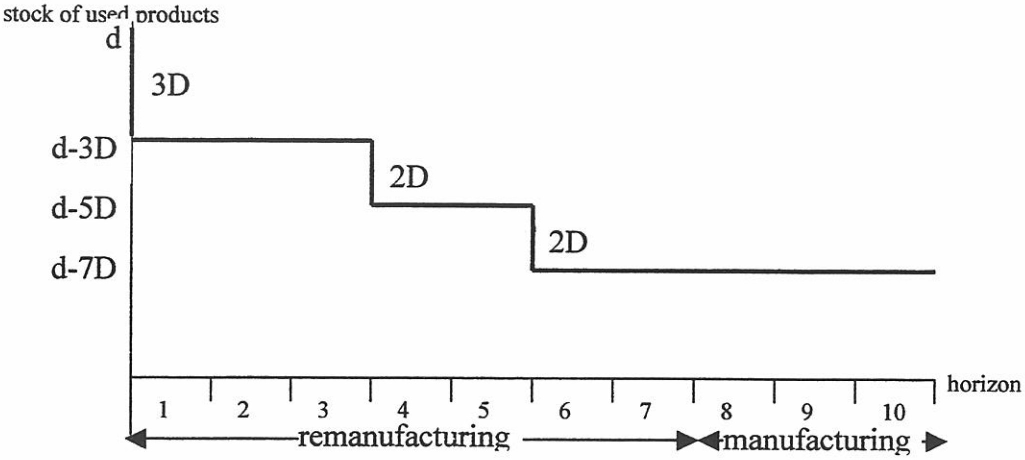

Figure 1. The stock level of used products for a problem with horizon $T = 10$, $n_1 = 3$ set up for remanufacturing, $n_2 = 2$ set up for manufacturing and switching period $k = 7$.

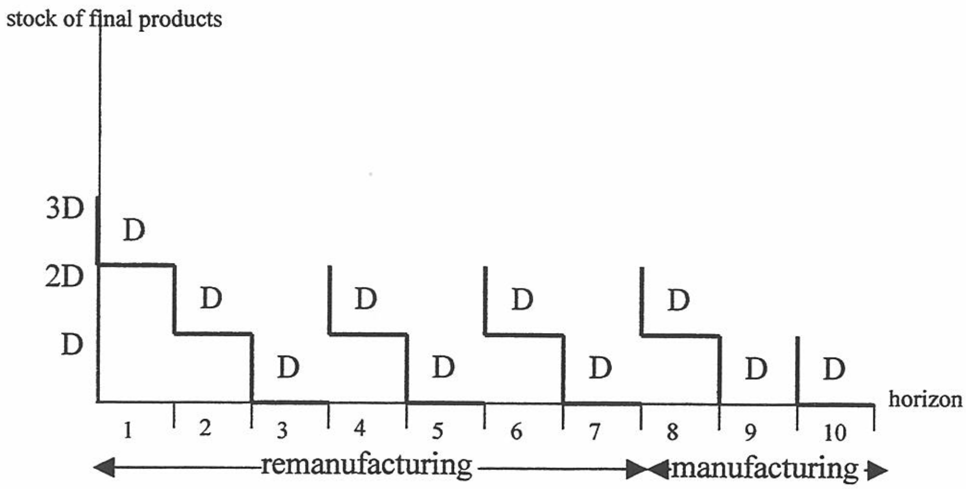

Figure 2. The stock level of final products for a problem with horizon $T = 10$, $n_1 = 3$ set up for remanufacturing, $n_2 = 2$ set up for manufacturing and switching period $k = 7$.

Although the used products balance equation for P_1 has been dropped, problems P_1 and $P_1^{'}$ are equivalent because, as we shall see any feasible solution $\{x_t, I_t\}$ generated by problem $P_1^{'}$ and extended by $y_t = d - tD - I_t$ is also feasible to problem P_1 (the reverse is obvious).

Lemma 1. The extended feasible solution of problem $P_1^{'}$ is also feasible for P_1.

Proof. From $y_t + I_t = d - tD, \quad t = 1,2,...,k$, we have

$$y_t = d - tD - I_t = d - tD - \left[\sum_{i=1}^{t} x_i - tD\right] = d - \sum_{i=1}^{t} x_i \geq TD - \sum_{i=1}^{t} x_i \geq 0.$$

The last inequality step is valid because the quantity of remanufactured products will not exceed the total demand. The y_t also fulfill the inventory balance equations,

$$y_t = d - tD - I_t = d - (t-1)D - D - I_{t-1} - x_t + D = y_{t-1} - x_t \qquad t = 2,3,...,k,$$

$$y_1 = d - D - I_1 = d - D - I_0 - x_1 + D = d - x_1. \ \square$$

Similarly the objective function of (6) becomes:

$$\sum_{t=k+1}^{T} \left(Sf(z_t) + hy_t + HI_t\right) = \sum_{t=k+1}^{T} \left(Sf(x_t) + HI_t\right) + (T-k)(d-kD)h$$

and problem P_2 is transformed to:

$$(P_2^{'}) \qquad \min_{z_t} \sum_{t=k+1}^{T} \left(Sf(z_t) + HI_t\right) + W^{'} \tag{11}$$

$$\text{s.t} \quad \begin{aligned} & I_t = I_{t-1} + z_t - D \\ & \sum_{t=k+1}^{T} f(z_t) = n_2 \\ & I_k = I_T = 0 \\ & z_t, y_t, I_t \geq 0, \qquad t = k+1, k+2, ..., T \end{aligned} \tag{12}$$

where $W^{'} = (T-k)(d-kD)h$.

Problems $P_1^{'}$ and $P_2^{'}$ are of the type studied by Papachristos and Ganas [4] and Chand [1]. The optimal policy to $P_1^{'}$, in the set of policies with n_1 remanufacturing set up is:

$x_1 = k - n_1\beta_1$ set up of type $a_1 = \left\lceil \frac{k}{n_1} \right\rceil$ and $y_1 = a_1 n_1 - k$ set up of type $\beta_1 = \left\lfloor \frac{k}{n_1} \right\rfloor$ in case $k \bmod n_1 \neq 0$. If $k \bmod n_1 = 0$ the above results to:

$x_1 = 0$ set up of type $\alpha_1 = \beta_1 + 1$ and $y_1 = n_1$ set up of type $\beta_1 = \frac{k}{n_1}$,

where by a *set up of type z* we mean one which produces a lot to cover exactly the demand for the next z periods.

The corresponding minimum cost is:

$$C^1(k,n_1) = n_1 R + \left[x_1 \frac{a_1(a_1-1)}{2} + y_1 \frac{\beta_1(\beta_1-1)}{2} \right] D(H-h) + kdh - \frac{k(k+1)}{2} hD. \qquad (13)$$

The optimal policy to P_2', in the set of policies with n_2 manufacturing set up is:

$x_2 = T - k - n_2\beta_2$ set up of type $a_2 = \left\lceil \frac{T-k}{n_2} \right\rceil$ and $y_2 = a_2 n_2 - T + k$ set up of type $\beta_2 = \left\lfloor \frac{T-k}{n_2} \right\rfloor$, in case $(T-k) \bmod n_2 \neq 0$. If $(T-k) \bmod n_2 = 0$ the above results to:

$x_2 = 0$ set up of type $\alpha_2 = \beta_2 + 1$ and $y_2 = n_2$ set up of type $\beta_2 = \frac{T-k}{n_2}$.

The corresponding minimum cost is:

$$C^2(k,n_2) = n_2 S + \left[x_2 \frac{a_2(a_2-1)}{2} + y_2 \frac{\beta_2(\beta_2-1)}{2} \right] DH + (T-k)(d-kD)h \qquad (14)$$

and

$$C_T(k,n_1,n_2) = C^1(k,n_1) + C^2(k,n_2) \qquad (15)$$

represents the optimal cost in the set of policies $P(n_1,n_2,k)$ and this is valid for any given $k \in N_k$.

3.2. Optimal policy in the set of policies $P(n_1,n_2,X)$

We shall now search for the optimal policy in this, wider than $P(n_1,n_2,k)$ class of policies, where X will be suitably defined subsets of N_k.

Let us consider the sets:

$$B_i = \{k : n_1 i \leq k < n_1(i+1),\ k \in N, i \in I\},$$

$$C_j = \{k : T - n_2(j+1) < k \leq T - n_2 j,\ k \in N, j \in J\},$$

$$A_{i,j} = B_i \cap C_j,$$

where $I = \left\{1,2,\ldots,\left\lceil \frac{T+1-n_1-n_2}{n_1} \right\rceil\right\}$, $J = \left\{1,2,\ldots,\left\lceil \frac{T+1-n_1-n_2}{n_2} \right\rceil\right\}$.

The collection of non empty sets $A_{i,j}$ constitutes a partition of N_k since $A_{i_0,j_0} \cap A_{i_1,j_1} = \varnothing\ (i_0 \neq i_1, j_0 \neq j_1)$ and $\cup_{\substack{i\in I\\ j\in J}} A_{i,j} = N_k$. So for any $k \in N_k$ there exist unique $i \in I$ and $j \in J$ such that, $k \in A_{i,j}$. The first non-empty $A_{i,j}$ set is $A_{i_{\max},1}$, where $i_{\max} = \left\lceil \frac{T+1-n_1-n_2}{n_1} \right\rceil$, while all other non–empty sets are derived by varying, increasing and decreasing, the indexes i and j. For any set $A_{i,j}$, we denote by $\max A_{i,j}$, $\min A_{i,j}$ its maximum and minimum elements respectively. If $n_0 = \min A_{i,j}$, $v_0 = \max A_{i-m,j+r}$ and $n_0 = v_0 + 1$ we say that $A_{i,j}$, $A_{i-m,j+r}$ are consecutive and more specifically, we say that $A_{i,j}$ is the predecessor of $A_{i-m,j+r}$ while $A_{i-m,j+r}$ is the successor set of $A_{i,j}$. For each pair of consecutive sets $A_{i,j}$ the following cases are possible. A set having exactly one element may have a consecutive one with exactly one element or with at least two elements and a set having at least two elements may have a consecutive one with exactly one element or with at least two elements. This observation will be useful in understanding the steps in the proof of theorem 2. Some properties of these sets are given in the appendix (A), while the example given in section 5 helps the understanding of their structure.

From the definition of $A_{i,j}$ we have that $i \le \frac{k}{n_1} < i+1$ and $j \le \frac{T-k}{n_2} < j+1$, for any $k \in A_{i,j} \neq \varnothing$. Also, for any $k \in A_{i,j} \neq \varnothing$, any of the following $k \bmod n_1 \neq 0$, $k \bmod n_1 = 0$, $(T-k) \bmod n_2 \neq 0$, $(T-k) \bmod n_2 = 0$ may be valid. Further, for any $i \in I$, $j \in J$ the intervals $[i, i+1)$, $[j, j+1)$ contain only the integers i and j. So for any $k \in A_{i,j} \neq \varnothing$ we shall always have:

$$\left\lfloor \frac{k}{n_1} \right\rfloor = i = \beta_1,\ a_1 = i+1,\ \left\lfloor \frac{T-k}{n_2} \right\rfloor = j = \beta_2,\ a_2 = j+1,$$

$$x_1 = k - n_1 i,\ y_1 = n_1(i+1) - k, \tag{16}$$

$$x_2 = T - k - n_2 j,\ y_2 = n_2(j+1) - T + k \text{ for any } k \in A_{i,j} \neq \varnothing.$$

Based on these relations, the expression (15) giving $C_T(k, n_1, n_2)$ becomes:

$$C_T(k, n_1, i, n_2, j) = n_1 R + \frac{iD(H-h)}{2}\left(2k - n_1(i+1)\right) + n_2 S + \frac{jDH}{2}\left(2(T-k) - n_2(j+1)\right)$$
$$+ Tdh - \frac{kDh}{2}\left(2T+1-k\right),\ \forall k \in A_{i,j} \neq \varnothing. \tag{17}$$

The indexes $i \in I$ and $j \in J$ were introduced into the cost function, just to indicate its strong dependence from them. The importance of these indexes is obvious, since they define the partition of N_k into the subsets $A_{i,j}$, so that for any $k \in A_{i,j}$ we have the simple expression (17) giving the cost of the optimal policy in the set of policies $P(n_1, n_2, k)$. Further to that, these indexes will play the key role in the search to establish stability regions. For any non empty set $A_{i,j}$ having at least two elements we have

$$P\left(n_1, n_2, A_{i,j}\right) = \bigcup_{k \in A_{i,j}} P(n_1, n_2, k).$$

We shall now search for the optimal policy in this $P\left(n_1, n_2, A_{i,j}\right)$, wider than $P(n_1, n_2, k)$, class of policies. This requires studying the function $C_T\left(k, n_1, i, n_2, j\right)$. The function $f(x)$ defined on a set X of consecutive integers is convex, if the difference function

$$\Delta f(x) = f(x) - f(x-1), \quad \forall x, x-1 \in \mathrm{X}$$

is increasing in x.

For any $k, k-1 \in A_{i,j}$ the difference function of $C_T\left(k, n_1, i, n_2, j\right)$ is:

$$\begin{aligned} \Delta A_{i,j}\left(k\right) &= \Delta C_T\left(k, n_1, i, n_2, j\right) = C_T\left(k, n_1, i, n_2, j\right) - C_T\left(k-1, n_1, i, n_2, j\right) \\ &= \left(i - j\right) HD - \left(T + i + 1 - k\right) hD. \end{aligned} \tag{18}$$

From (18) it is obvious that for $i \le j$ the difference $\Delta A_{i,j}\left(k\right)$ is always negative and increasing in k, and so $C_T\left(k, n_1, i, n_2, j\right)$ is decreasing and convex on every such $A_{i,j}$ set. For $i > j$, $\Delta A_{i,j}\left(k\right)$ is again increasing and so $C_T\left(k, n_1, i, n_2, j\right)$ is again convex, decreasing for all $k \in A_{i,j}$ such that $\frac{H}{h} < \frac{T+i+1-k}{i-j}$ and increasing, for all $k \in A_{i,j}$ such that $\frac{H}{h} > \frac{T+i+1-k}{i-j}$. Moreover $\Delta A_{i,j}\left(k\right)$ vanishes, if there exist a k such that $\frac{H}{h} = \frac{T+i+1-k}{i-j}$.

The above discussion leads to the following theorem.

Theorem 1. For any set $A_{i,j}$ having at least two elements, the optimal policy in the class of policies $P\left(n_1, n_2, A_{i,j}\right)$ is the one with k_{opt} defined as following:

1. If $i \leq j$, then always $k_{opt} = \max A_{i,j}$.
2. If $i > j$ then:
   a) For $k_0 - 1, k_0, k_0 + 1 \in A_{i,j}$ and k_0 such that
   $$\frac{T+i+1-(k_0+1)}{i-j} < \frac{H}{h} < \frac{T+i+1-k_0}{i-j},\ k_{opt} = k_0.$$
   b) For $k_0 - 1, k_0 \in A_{i,j}$, such that $\dfrac{H}{h} = \dfrac{T+i+1-k_0}{i-j}$, $k_{opt} = k_0$ or $k_{opt} = k_0 - 1$.
   c) If for all $k-1, k \in A_{i,j}$, $\Delta A_{i,j}(k) > 0$, i.e. $\dfrac{H}{h} > \dfrac{T+i+1-k}{i-j}$ then $k_{opt} = \min A_{i,j}$.
   d) If for all $k-1, k \in A_{i,j}$, $\Delta A_{i,j}(k) < 0$, i.e. $\dfrac{H}{h} < \dfrac{T+i+1-k}{i-j}$ then $k_{opt} = \max A_{i,j}$.

In either case, the type and number of set up are determined by relations (16) and the cost by (17). □

This theorem is useful by itself because it determines the optimal policy in the class of policies $P\left(n_1, n_2, A_{i,j}\right)$, only with reference to the values of the ratio $\dfrac{H}{h}$. Moreover it is easy to see what is the stability region of the so obtained policy. However, at present, we skip the stability issue as we shall come back to this with detailed discussion at section 4. The application of this theorem is illustrated by, the example given in section 5.

3.3. Convexity of the total cost function and overall optimal policy

It is obvious that the set of all admissible policies is

$$P\left(n_1, n_2, N_k\right) = \cup_{\substack{i \in I \\ j \in J}} P\left(n_1, n_2, A_{i,j}\right)$$

and so the search for the overall optimal must be done within the set of policies $P(n_1, n_2, N_k)$. Further, if we denote by $C_T(n_1, n_2)$ the optimal cost for problem P in the set of policies $P(n_1, n_2, N_k)$, this is obtained from:

$$C_T(n_1, n_2) = \min_{k \in N_k}\left(C_T(k, n_1, i, n_2, j)\right).$$

We have already seen that $C_T(k,n_1,i,n_2,j)$ is convex on every $A_{i,j}$. To search for convexity of $C_T(k,n_1,i,n_2,j)$ on the set N_k, we need to have its difference function for any $k,k-1\in N_k$ and especially for the case that $k\in A_{i,j}$ while $k-1\in A_{i-m,j+r}$, where $A_{i,j}$, $A_{i-m,j+r}$ are consecutive sets. To distinguish this difference from $\Delta A_{i,j}(k)$ defined on $A_{i,j}$, we call it the jump of $C_T(k,n_1,i,n_2,j)$ between the consecutive sets $A_{i,j}$, $A_{i-m,j+r}$ and we use the symbol $J\left(A_{i,j},A_{i-m,j+r}\right)$ to represent it. This jump is:

$$J\left(A_{i,j},A_{i-m,j+r}\right)=C_T\left(n_0,n_1,i,n_2,j\right)-C_T\left(v_0,n_1,i-m,n_2,j+r\right) \tag{19}$$

where $n_0=\min A_{i,j}$ and $v_0=\max A_{i-m,j+r}$.

Substituting $C_T(k,n_1,i,n_2,j)$ from (17), we have:

$$\begin{aligned}
&J\left(A_{i,j},A_{i-m,j+r}\right)=\\
&\quad=\frac{D(H-h)}{2}\left[2\left(i+v_0m\right)-mn_1\left(2i+1-m\right)\right]\\
&\quad+\frac{DH}{2}\left[n_2r\left(2j+r+1\right)-2\left(Tr+j-v_0r\right)\right]-hD\left(T-v_0\right)\\
&\quad=\frac{DH}{2}\left[2\left(i+v_0m\right)-mn_1\left(2i+1-m\right)+n_2r\left(2j+r+1\right)-2\left(Tr+j-v_0r\right)\right]\\
&\quad-\frac{hD}{2}\left[2\left(i+v_0m\right)-mn_1\left(2i+1-m\right)+2\left(T-v_0\right)\right]
\end{aligned} \tag{20}$$

If we take into account the values of indexes m,r defining consecutive sets (appendix A), from (20) it is easy to prove that:

- For all pairs of consecutive sets $A_{i,j}$, $A_{i-m,j+r}$ with $i-m<j+r$ we have $J\left(A_{i,j},A_{i-m,j+r}\right)<0$ irrespectively of whether $i\le j$, or $i>j$.
- For all sets $A_{i,j}$ with $i>j$ and its successor $A_{i-m,j+r}$ with $i-m=j+r$, we have $J\left(A_{i,j},A_{i-m,j+r}\right)<0$, in case where $m=1$ and $r=0$. In case where $m=0$ and $r=1$ or $m=1$ and $r=1$, the sign of $J\left(A_{i,j},A_{i-m,j+r}\right)$ depends on the ratio $\frac{H}{h}$.

- For all sets $A_{i,j}$ with $i>j$ and its successor $A_{i-m,j+r}$ with $i-m>j+r$, the sign of $J\left(A_{i,j},A_{i-m,j+r}\right)$ depends on the ratio $\frac{H}{h}$. Properties established for $A_{i,j}$ ensure that, in this case, we need to check only the values 0 or 1 for the parameters m,r.

We are now ready to establish the following.

Theorem 2. The function $C_T\left(k,n_1,i,n_2,j\right)$, $k\in N_k$, is convex with respect to *k*.

Proof. An outline is given in the appendix (B). □

The convexity of the $C_T\left(k,n_1,i,n_2,j\right)$ over N_k guarantees the existence of a global minimum, which obviously occurs at the point k_0 where the difference function $\Delta C_T\left(k,n_1,i,n_2,j\right)$ changes sign, if it changes, from negative to positive values.

We have seen previously that $\Delta A_{i,j}(k)<0$ and $J\left(A_{i,j},A_{i-m,j+r}\right)<0$ for certain combinations of the indexes i,j. So again from (18), (20) it is obvious that the determination of k_{opt} requires to compute the following:

$$S\left(A_{i,j}^k\right)=\frac{H}{h}-\frac{T+i+1-k}{i-j}=L-\delta_{i,j}^k,\ \text{for } i>j \text{ and all } k,k-1\in A_{i,j} \tag{21}$$

and

$$\begin{aligned}S\left(A_{i,j},A_{i-m,j+r}\right)&=\frac{H}{h}-\frac{2\left(i+v_0m\right)-mn_1\left(2i+1-m\right)+2\left(T-v_0\right)}{2\left(i+v_0m\right)-mn_1\left(2i+1-m\right)+n_2r\left(2j+r+1\right)-2\left(Tr+j-v_0r\right)}\\&=L-j_{i,j/i-m,j+r}\end{aligned} \tag{22}$$

for all pairs of consecutive sets $A_{i,j}$, $A_{i-m,j+r}$ with $i>j$ and $i-m>j+r$. In case that $i-m=j+r$, it is necessary to compute $S\left(A_{i,j},A_{i-m,j+r}\right)$ only for $m=0$ and $r=1$ or $m=1$ and $r=1$ (for $m=1$ and $r=0$ the jump is negative). Based on the previous discussion we propose the algorithm, which computes the optimal policy and constructs its stability region.

4. The algorithm which computes the optimal policy and constructs its stability regions

In this section we present the algorithm which determines the optimal policy for any value of the ratio $L = \frac{H}{h}$ and constructs its corresponding stability region. The steps of the algorithm are:

Step 1. Construct the sets B_i and C_j.

Step 2. Take the intersections $A_{i,j} = B_i \cap C_j$. The first non-empty $A_{i,j}$ set is $A_{i_{\max},1}$.

Step 3. For any pair of consecutive sets $A_{i,j}, A_{i-m,j+r}$ determine $n_0 = \min A_{i,j}$, $\nu_0 = \max A_{i-m,j+r}$ and the values for m and r needed in (22) (the properties established for $A_{i,j}$, ensure that m,r can take only any of the values 0 or 1).

Step 4. For all suitable $i > j$ determine the numbers $\delta_{i,j}^k$, appeared in (21). Also for all suitable $i > j$ and $i+m \geq j-r$, the numbers $j_{i,j/i-m,j+r}$ appeared in (22) (for reasons of better understudying, we have inserted the sign functions instead of these numbers in Table 2).

Step 5. Arrange in increasing order the numbers 1, $\delta_{i,j}^k$, $j_{i,j/i-m,j+r}$, ∞ obtained in step 4 (1 and ∞ are included because these are the two extreme values for the parameter L). Starting from 1 create all the intervals by taking pairs of successive numbers. The limits of these intervals are obviously depended from the sets $A_{i,j}$, and may correspond to pairs of numbers of any of the following types:

1. $\left(1, \delta^k{}_{i_{\max},1}\right)$, when $k = \max A_{i_{\max},1}$ and $A_{i_{\max},1}$ has at least two elements.
2. $\left(1, j_{i_{\max},1/i_{\max}-m,r+1}\right)$, when $A_{i_{\max},1}$ has exactly one element.
3. $\left(\delta_{i,j}^k, j_{i,j/i-m,j+r}\right)$, where $k = \min A_{i,j} + 1$.
4. $\left(j_{i,j/i-m,j+r}, \delta_{i-m,j+r}^k\right)$, where $k = \max A_{i-m,j+r}$.
5. $\left(\delta_{i,j}^{k+1}, \delta_{i,j}^k\right)$, where $k-1,\ k,\ k+1 \in A_{i,j}$.
6. $\left(j_{i+m,j-r/i,j}, j_{i,j/i-m,j+r}\right)$, where $A_{i,j}$ has one element.

7. $\left(\delta^k_{i,j},\infty\right)$ where $A_{i,j}$ is the last non empty set with $i>j$, starting the scaning from $A_{i_{\max},1}$ and has at least two elements, while $k=\min A_{i,j}+1$.

8. $\left(j_{i,j/i-m,j+r},\infty\right)$, where $A_{i-m,j+r}$ is the last non empty set with $i-m=j+r$, and $m=0$, $r=1$ or $m=1$, $r=1$.

The sets $A_{i+m,j-r}$, $A_{i,j}$, $A_{i-m,j+r}$ are non-empty consecutive.

If the ratio L belongs to any of the above defined intervals, the optimum solution is easily determined and the interval is its corresponding *stability region*. So we have.

Step 6. For any of the previously calculated stability regions, the k_{opt} value for k is:

1. For a stability region $\left(1,\delta^k{}_{i_{\max},1}\right)$ or $\left(1,j_{i_{\max},1/i_{\max}-m,r+1}\right)$, $k_{opt}=\max A_{i_{\max},1}$ or $k_{opt}=\min A_{i_{\max},1}$ respectively (note that only one of the two cases may appear and the second case appears only if $A_{i_{\max},1}$ has exactly one element).
2. For any stability region $\left(\delta^k_{i,j},j_{i,j/i-m,j+r}\right)$, $k_{opt}=\min A_{i,j}$.
3. For any stability region $\left(j_{i,j/i-m,j+r},\delta^k_{i-m,j+r}\right)$, $k_{opt}=\max A_{i-m,j+r}$.
4. For any stability region $\left(\delta^{k+1}_{i,j},\delta^k_{i,j}\right)$, $k_{opt}=k$.
5. For any stability region $\left(j_{i+m,j-r/i,j},j_{i,j/i-m,j+r}\right)$, $k_{opt}=\min A_{i,j}$.
6. For a stability region $\left(\delta^k_{i,j},\infty\right)$ or $\left(j_{i,j/i-m,j+r},\infty\right)$, $k_{opt}=\min A_{i,j}$ or $k_{opt}=\max A_{i-m,j+r}$ respectively (only one of the two cases may appear).
7. We may also have cases that k satisfies the following:
   a) $\dfrac{H}{h}=\delta^k{}_{i,j}$. If so then $k_{opt}=k$ or $k_{opt}=k-1$.
   b) $\dfrac{H}{h}=j_{i,j/i-m,j+r}$. If so then $k_{opt}=\min A_{i,j}$ or $\max A_{i-m,j+r}$.

Step 7. Use the k_{opt} obtained in step 6 and determine the optimal policy using (16).

The results of theorems 1 and 2 guarantee that the $\dfrac{H}{h}$ intervals determined in Step 5 are stability regions for the k_{opt} values calculated in Step 6.

5. Numerical Example

We consider a problem with $T=30$, $n_1=5$ and $n_2=4$.

The sets B_i, C_j and $A_{i,j}$ are:

B_1={5,6,7,8,9}

B_2={10,11,12,13,14}

B_3={15,16,17,18,19}

B_4={20,21,22,23,24}

B_5={25,26,27,28,29}

C_1={23,24,25,26}

C_2={19,20,21,22}

C_3={15,16,17,18}

C_4={11,12,13,14}

C_5={7,8,9,10}

C_6={3,4,5,6}

$A_{1,6}=B_1\cap C_6$={5,6}

$A_{1,5}=B_1\cap C_5$={7,8,9}

$A_{2,5}=B_2\cap C_5$={10}

$A_{2,4}=B_2\cap C_4$={11,12,13,14}

$A_{3,3}=B_3\cap C_3$={15,16,17,18}

$A_{3,2}=B_3\cap C_2$={19}

$A_{4,2}=B_4\cap C_2$={20,21,22}

$A_{4,1}=B_4\cap C_1$={23,24}

$A_{5,1}=B_5\cap C_1$={25,26}

Let us take the sets $A_{2,4}$={11,12,13,14} and $A_{4,2}$={20,21,22} to illustrate the application of theorem 1. For the set $A_{2,4}$, we compute the difference functions which are:

$\Delta A_{2,4}(12)=C_{30}(12,5,2,4,4)-C_{30}(11,5,2,4,4)=-4HD-21hD<0$,

$\Delta A_{2,4}(13)=-4HD-20hD<0$ and $\Delta A_{2,4}(14)=-4HD-19hD<0$. So the optimum in the class of policies $P(n_1=5, n_2=4, A_{2,4})$ is $k_{opt}=\max A_{2,4}=14$.

For the set $A_{4,2}$, we compute $\Delta A_{4,2}(21)=2HD-14hD=2hD\left(\frac{H}{h}-7\right)$ and

$\Delta A_{4,2}(22)=2HD-13hD=2hD\left(\frac{H}{h}-\frac{13}{2}\right)$. For any $\frac{H}{h}\in\left(\frac{13}{2},7\right)$, the optimal policy has $k_{opt}=21$ and the interval $\left(\frac{13}{2},7\right)$ is its stability region. If $\frac{H}{h}=7$, then $k_{opt}=21$ or $k_{opt}=20$.

We shall now explain how to find the optimal policy and its corresponding stability region for any set of values H,h having a ratio $L=\frac{H}{h}$.

Applying steps 3 and 4 of the algorithm we compute $\delta_{i,j}^{k}$ on each of the sets $A_{4,2}, A_{4,1}, A_{5,1}$ and similarly the $j_{i,j/i-m,j+r}$ for all consecutive pairs of sets, $A_{5,1}, A_{4,1}, A_{4,2}, A_{3,2}, A_{3,3}$. So we have:

$S\left(A_{5,1}^{26}\right) = L - \delta_{5,1}^{26} = L - \frac{10}{4}$, $S\left(A_{5,1}, A_{4,1}\right) = L - j_{5,1/4,1} = L - \frac{10}{3}$,

$S\left(A_{4,1}^{24}\right) = L - \delta_{4,1}^{24} = L - \frac{11}{3}$, $S\left(A_{4,1}, A_{4,2}\right) = L - j_{4,1/4,2} = L - 4$,

$S\left(A_{4,2}^{22}\right) = L - \delta_{4,2}^{22} = L - \frac{13}{2}$, $S\left(A_{4,2}^{21}\right) = L - \delta_{4,2}^{21} = L - 7$,

$S\left(A_{4,2}, A_{3,2}\right) = L - j_{4,2/3,2} = L - 14$ and $S\left(A_{3,2}, A_{3,3}\right) = L - j_{3,2/3,3} = L - 15$.

The stability regions are:

$\left(1, \frac{10}{4}\right)$, $\left(\frac{10}{4}, \frac{10}{3}\right)$, $\left(\frac{10}{3}, \frac{11}{3}\right)$, $\left(\frac{11}{3}, 4\right)$, $\left(4, \frac{13}{2}\right)$, $\left(\frac{13}{2}, 7\right)$, $(7,14)$, $(14,15)$, $(15, \infty)$.

Now for any L interval we apply step 6 to find k_{opt} . Example for $L \in \left(1, \frac{10}{4}\right)$, $k_{opt} = 26$, while for $L \in \left(\frac{10}{4}, \frac{10}{3}\right)$, $k_{opt} = 25$. Using the same procedure, we can calculate all other k_{opt} values. The results are given in Table 2.

The cost function of the above problem with $H = 10$, $h = 2$, $d = 30$, $D = 1$, $R = 5$ and $S = 3$ is given in Figure 3.

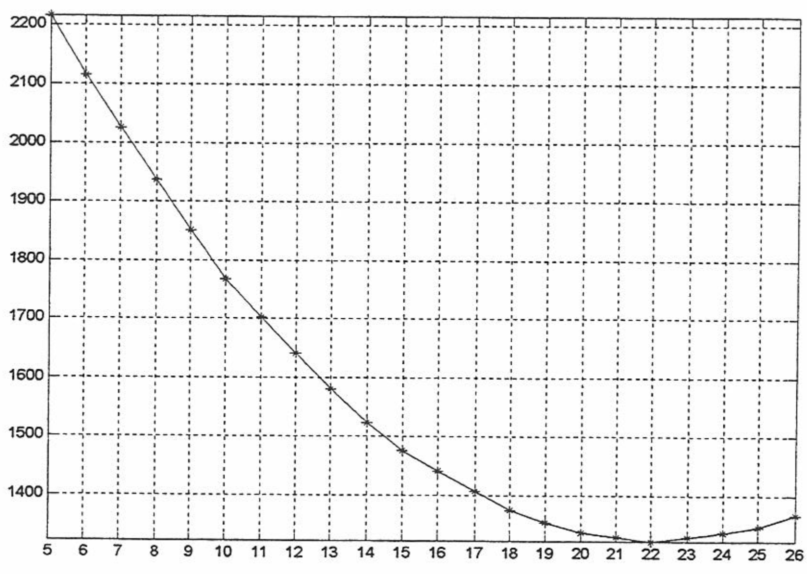

Figure 3. Cost function for a problem with $T = 30, n_1 = 5, n_2 = 4, H = 10, h = 2, d = 30, D = 1, R = 5, S = 3$.

6. Conclusion

In this paper, we have studied a reverse Wagner/Whitin type production and inventory control model. For this model, we supposed that cost and demand parameters are constant and a sufficiently large quantity of used products is available at the beginning of the planning horizon. We considered policies with given number of set up for remanufacturing and manufacturing respectively. In this class of policies we found the optimal policy, which specifies the number of periods where demand is covered either only by remanufactured items or by newly manufactured items respectively, the periods where remanufacturing or manufacturing activities will take place and the corresponding quantities. Further we constructed stability regions for the optimal policy, expressed as intervals of the ratio of holding cost parameters, which is the main objective of the paper.

Although stability issues are quite difficult and hard problems it is hoped that further research on the following can give some results. Consider policies with variable numbers of set up for manufacturing, remanufacturing and try to obtain stability regions for the so obtained optimal policy. The model can be extended to include variable manufacturing remanufacturing set up cost. Although such models are much more complicated there is some evidence that similar stability results can be obtained. The case, where the initial stock of used products is not enough to cover the total demand for the whole horizon, still remains to be examined as it is also noted in [7].

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Appendix A. Properties of sets $A_{i,j}$

We shall present here the properties of the sets $A_{i,j}$. The important question is which of the $A_{i,j}$ are non empty. Let us suppose that $n_1 > n_2$ i.e. the number of set-up for remanufacturing is greater than the number of set-up for manufacturing. Then the following are true:

1. For any i, j satisfying the relation $n_1 i + n_2 j \le T - n_1 - n_2 + 1$, we have $A_{i,j} = \varnothing$.
2. For any i, j satisfying the relation $T - n_1 - n_2 + 1 < n_1 i + n_2 j < T - n_1 + 1$, we have $A_{i,j} \ne \varnothing$.
3. For any indexes i, j satisfying the relation $T - n_1 + 1 \le n_1 i + n_2 j \le T - n_2 + 1$, we have $C_j \subseteq B_i$ and so $A_{i,j} = C_j$ and has exactly n_2 elements.
4. For any indexes i, j satisfying the relation $T - n_2 + 1 < n_1 i + n_2 j \le T$, we have $A_{i,j} \ne \varnothing$.
5. For any indexes i, j satisfying the relation $T < n_1 i + n_2 j$, we have $A_{i,j} = \varnothing$.

Based on the above properties we have the following results which characterize the non empty consecutive sets.

- If $A_{i,j} \ne \varnothing$ and the relation $T - n_1 - n_2 + 1 < n_1 i + n_2 j < T - n_1 + 1$ is fulfilled, then the non empty successor to $A_{i,j}$ set is the $A_{i,j+1}$.
- If $A_{ij} \ne \varnothing$ and the relation $T - n_1 + 1 \le n_1 i + n_2 j < T - n_2 + 1$ is fulfilled, then the non empty successor to $A_{i,j}$ set is the $A_{i,j+1}$. If $n_1 i + n_2 j = T - n_2 + 1$, then the non empty successor to $A_{i,j}$ set is the $A_{i-1,j+1}$.
- If $A_{ij} \ne \varnothing$ and the relation $T - n_2 + 1 < n_1 i + n_2 j \le T$ is fulfilled, then the non empty successor to $A_{i,j}$ set is the $A_{i-1,j}$.

These are the only cases where non empty successor to $A_{i,j}$ sets are defined.

Let us now suppose that $n_1 \le n_2$. Then the following are true:

a) For any i, j satisfying the relation $n_1 i + n_2 j \le T - n_1 - n_2 + 1$, we have $A_{i,j} = \varnothing$.

b) For any i, j satisfying the relation $T - n_1 - n_2 + 1 < n_1 i + n_2 j < T - n_2 + 1$, we have $A_{i,j} \ne \varnothing$.

c) For any indexes i, j satisfying the relation $T - n_2 + 1 \le n_1 i + n_2 j \le T - n_1 + 1$, we have $B_i \subseteq C_j$ and so $A_{i,j} = B_i$ and has exactly n_1 elements.

d) For any indexes i, j satisfying the relation $T - n_1 + 1 < n_1 i + n_2 j \le T$, we have $A_{i,j} \ne \varnothing$.

e) For any indexes i, j satisfying the relation $T < n_1 i + n_2 j$, we have $A_{i,j} = \varnothing$.

Based on the above properties we have the following results which characterize the non empty consecutive sets.

- If $A_{i,j} \ne \varnothing$ and the relation $T - n_1 - n_2 + 1 < n_1 i + n_2 j < T - n_2 + 1$ is fulfilled, then the non empty successor to $A_{i,j}$ set is the $A_{i,j+1}$.
- If $A_{ij} \ne \varnothing$ and the relation $T - n_2 + 1 < n_1 i + n_2 j \le T - n_1 + 1$ is fulfilled, then the non empty successor to $A_{i,j}$ set is the $A_{i-1,j}$. If $n_1 i + n_2 j = T - n_2 + 1$, then the non empty successor to $A_{i,j}$ set is the $A_{i-1,j+1}$.
- If $A_{ij} \ne \varnothing$ and the relation $T - n_1 + 1 < n_1 i + n_2 j \le T$ is fulfilled, then the non empty successor to $A_{i,j}$ set is the $A_{i-1,j}$.

Appendix B. Proof of theorem 2

To prove the convexity of $C_T(k, n_1, i, n_2, j)$, we must establish that its first difference is an increasing function of k. Checking carefully all possible cases that may occur for the sets $A_{i,j}$ and between consecutive pairs of them, we result that this requires to prove the following:

First:

$$\Delta A_{i,j}(k) > J(A_{i,j}, A_{i,j+1}),\ \ \Delta A_{i,j}(k) > J(A_{i,j}, A_{i-1,j}),\ \ \Delta A_{i,j}(k) > J(A_{i,j}, A_{i-1,j+1})$$

for any $A_{i,j}$ having at least two elements and its possible successor non empty sets $A_{i,j+1}$, $A_{i-1,j}$, $A_{i-1,j+1}$. The difference $\Delta A_{i,j}(k)$ is calculated at $k = \min A_{i,j} + 1$.

The above relations ensure that any difference is greater than the jump which may follows immediately.

Second:

$$J(A_{i+1,j}, A_{i,j}) > \Delta A_{i,j}(k), J(A_{i,j-1}, A_{i,j}) > \Delta A_{i,j}(k), J(A_{i+1,j-1}, A_{i,j}) > \Delta A_{i,j}(k)$$

for any $A_{i,j} \neq \varnothing$ set having at least two elements and its possible predecessor sets $A_{i+1,j}$, $A_{i,j-1}$, $A_{i+1,j-1}$. The difference $\Delta A_{i,j}(k)$ is calculated at $k = \max A_{i,j}$.

The above relations ensure that any jump is greater than the difference which may follow immediately.

Third:

1. $J(A_{i+1,j}, A_{i,j}) > J(A_{i,j}, A_{i-1,j})$
2. $J(A_{i+1,j-1}, A_{i,j}) > J(A_{i,j}, A_{i-1,j})$
3. $J(A_{i,j-1}, A_{i,j}) > J(A_{i,j}, A_{i-1,j})$
4. $J(A_{i+1,j}, A_{i,j}) > J(A_{i,j}, A_{i,j+1})$
5. $J(A_{i+1,j-1}, A_{i,j}) > J(A_{i,j}, A_{i,j+1})$
6. $J(A_{i,j-1}, A_{i,j}) > J(A_{i,j}, A_{i,j+1})$
7. $J(A_{i+1,j}, A_{i,j}) > J(A_{i,j}, A_{i-1,j+1})$
8. $J(A_{i+1,j-1}, A_{i,j}) > J(A_{i,j}, A_{i-1,j+1})$
9. $J(A_{i,j-1}, A_{i,j}) > J(A_{i,j}, A_{i-1,j+1})$

for any set $A_{i,j}$ having exactly one element and all other sets involved in the above inequalities having at least one element. All pairs of sets appeared in parenthesis in the above relations are consecutive.

The above relations ensure that any jump is greater than the jump which may follows immediately.

We shall prove only the following three inequalities:

$\Delta A_{i,j}(k) > J(A_{i,j}, A_{i,j+1})$, $J(A_{i+1,j}, A_{i,j}) > \Delta A_{i,j}(k)$ and $J(A_{i,j-1}, A_{i,j}) > J(A_{i,j}, A_{i-1,j})$.

The others are proved similarly.

Using (18) and (20), $\Delta A_{i,j}(k) > J(A_{i,j}, A_{i,j+1})$ becomes:

$$(i-j)HD-(T+i+1)hD+(n_0+1)hD>i(H-h)D-(T+j-v_0-n_2(j+1))HD-(T-v_0)hD,$$

where $n_0 = \min A_{i,j}$ and $v_0 = \max A_{i,j+1}$. From the above relation we obtain

$$H(T-v_0-n_2(j+1))+h>0.$$

This is valid because $v_0 = \max A_{i,j+1}$ and from the definition of $A_{i,j+1}$ it follows $v_0 \le T - n_2(j+1)$.

Similarly $J(A_{i+1,j}, A_{i,j}) > \Delta A_{i,j}(k)$ gives:

$$(i+1+v_0-n_1(i+1))(H-h)D - jHD - (T-v_0)hD > (i-j)HD - (T+i+1-v_0)hD,$$

where $n_0 = \min A_{i+1,j}$ and $v_0 = \max A_{i,j}$. From this we have

$$H(v_0+1-n_1(i+1)) - h(v_0-n_1(i+1)) > 0 \Rightarrow$$
$$H(n_0-n_1(i+1)) - h(v_0-n_1(i+1)) > 0.$$

But since $n_0 = \min A_{i+1,j}$ and $v_0 = \max A_{i,j}$, we have $n_0 - n_1(i+1) \ge 0$ and $v_0 - n_1(i+1) < 0$, which establishes the inequality.

Using (18), $J(A_{i,j-1}, A_{i,j}) > J(A_{i,j}, A_{i-1,j})$ becomes:

$$\begin{aligned} i(H-h)D+(n_2j-T-j+v_1)HD-(T-v_0)hD &> \\ >(i+n_0-n_1i)(H-h)D-jHD-(T-n_0)hD & \end{aligned}$$

where $n_0 = \max A_{i-1,j}$, $A_{i,j} = \{v_0\}$ and $v_1 = \min A_{i,j-1}$. Form this we have

$$H(v_1 - T + n_2j + n_1i - n_0) + h(v_0 - n_1i) > 0.$$

But, since $n_0 = \max A_{i-1,j}$, $A_{i,j} = \{v_0\}$ and $v_1 = \min A_{i,j-1}$, we have $n_1i - n_0 > 0$, $v_1 - T + n_2j > 0$ and $v_0 - n_1i \ge 0$. □

Table 2. Solution table for $T = 30,\ n_1 = 5,\ n_2 = 4$.

i,j	$A_{i,j}$	v_0	n_0	m,r	$S(A_{i,j}) = L - \delta_{i,j}^k$	$S(A_{i,j}, A_{i-m,i+r}) =$ $L - j_{i,j/i-m,j+r}$	$L = \frac{H}{h}$ stab. reg	k_{opt}	$\frac{H}{h}$	k_{opt}
3,3	{15,16,17,18}	18			It is always negative					
				0,1	-	$L - j_{3,2/3,3} = L - 15$	$15 < L$	18	$L = 15$	18,19
3,2	{19}	19	19		-					
				1,0		$L - j_{4,2/3,2} = L - 14$	$14 < L < 15$	19	$L = 14$	19,20
4,2	{20,21,22}	22	20		$L - \delta_{4,2}^{21} = L - 7$		$7 < L < 14$	20	$L = 7$	20,21
					$L - \delta_{4,2}^{22} = L - 13/2$		$13/2 < L < 7$	21	$L = 13/2$	21,22
				0,1		$L - j_{4,1/4,2} = L - 4$	$4 < L < 13/2$	22	$L = 4$	22,23
4,1	{23,24}	24	23		$L - \delta_{4,1}{}^{24} = L - 11/3$		$11/3 < L < 4$	23	$L = 11/3$	23,24
				1,0		$L - j_{5,1/4,1} = L - 10/3$	$10/3 < L < 11/3$	24	$L = 10/3$	24,25
5,1	{25,26}	26	25		$L - \delta_{5,1}^{26} = L - 10/4$		$10/4 < L < 10/3$	25	$L = 10/4$	25,26
							$1 < L < 10/4$	26		